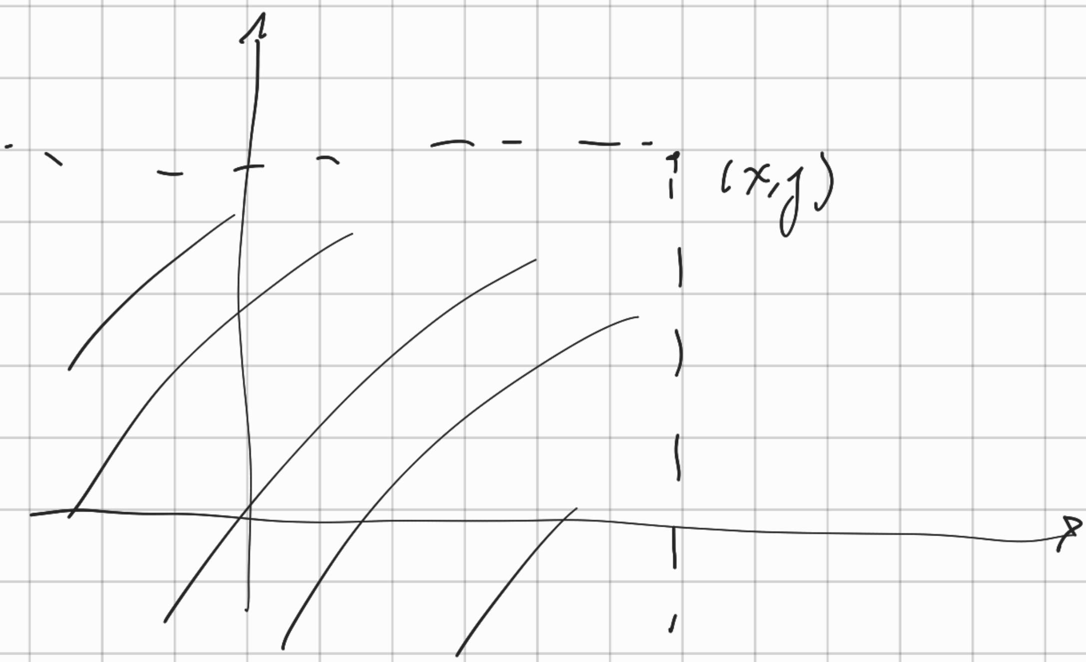
Probability Theory

10/13/22

Jointly distributed cont. r.v.

$F(x) = \mathbb{P}(X \leq x)$ 1 r.v.

$F(x, y) = \mathbb{P}(X \leq x \ \& \ Y \leq y)$

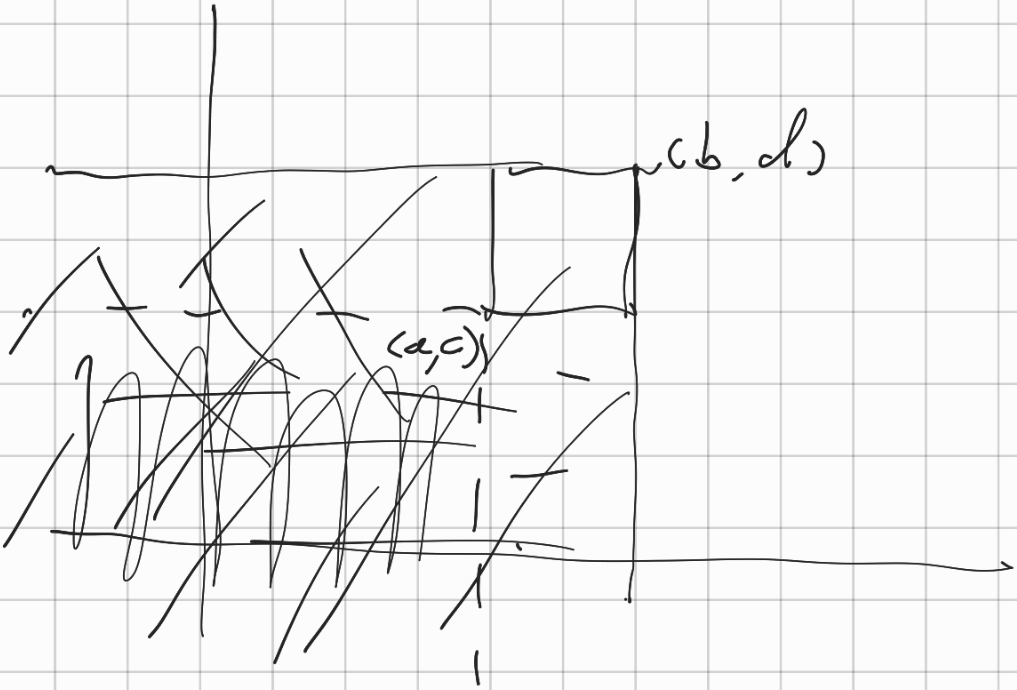


$F(x,y)$

$\mathbb{P}(a < X \leq b \text{ and } c < Y \leq d) =$

$F(b,d) - F(b,c) - F(a,d) +$

$F(a,c)$



Once you have the prob. of rectangles,

X and Y are jointly continuous if $F(x,y)$ is differentiable.

$$\mathbb{P}(x < X < x + dx,\ y < Y < y + dy) = f(x,y)\, dx\, dy$$

$$f(x,y) = \partial_x \partial_y F(x,y)$$

$$F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)\, du\, dv$$

Uniform dist.

X, Y are uniform in

$[0,1] \times [0,1]$

$$f(x,y) = \begin{cases} 1 & \text{if } 0 < x, y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f(x,y) = f_X(x) \, f_Y(y)$$

$$f_X(x) = 1 \quad \text{if } 0 \leq x \leq 1$$

X and Y are independent.

Uniform in $[A, B] \times [C, D]$

$$f(x,y) = \frac{1}{(B-A) \cdot (D-C)} \qquad \begin{matrix} A \leq x \leq B \\ C \leq y \leq D \end{matrix}$$

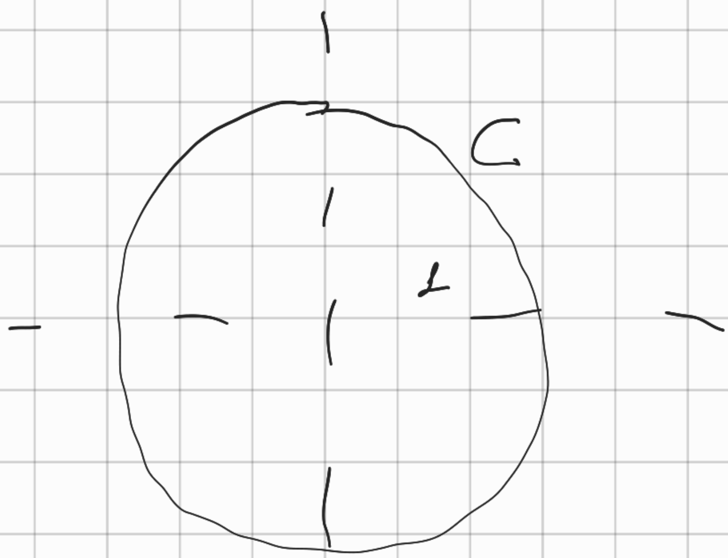
if A is a subset of $\mathbb{R}^2$ (e.g. a circle) uniform in A

$$f(x,y) = \begin{cases} \frac{1}{|A|} & \text{if } (x,y) \in A \\ 0 & \text{otherwise} \end{cases}$$

In the case of rectangle

$f(x,y) = f_X(x)\, f_Y(y) \Rightarrow$ indep.

In the case of circle $\Rightarrow$ not indep.



$|C| = \pi$

$$f(x,y) = \begin{cases} c & \text{if } (x,y) \in C \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 = c\,|C|$$

Rectangle

$$f(x, y) = g(x)\, h(y)$$

$$g(x) = \frac{1}{(B-A)(C-D)} \qquad A \leq x \leq B$$

$$h(y) = 1 \qquad C \leq y \leq D$$

Marginals

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

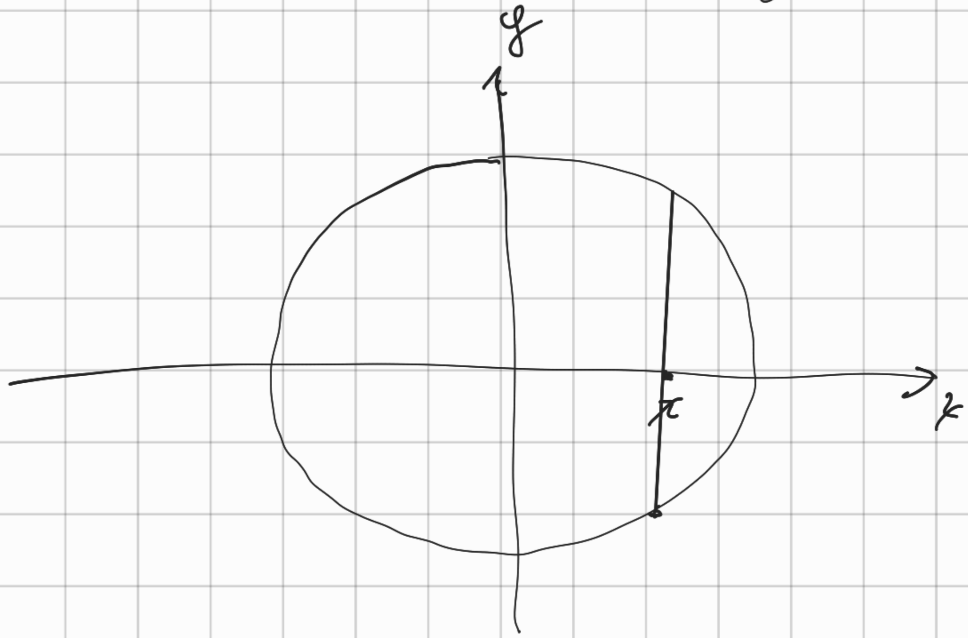
$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

Uniform in $[A, B] \times [C\ D]$

$$f_X(x) = \frac{1}{B-A} \qquad A \leq x \leq B$$

$$f_Y(y) = \frac{1}{D-C} \qquad C \leq x \leq D$$

Uniform on a circle of radius 1 cent. at the origin



$$f_X(x) = 2\sqrt{1-x^2} \qquad -1 \le x \le 1$$

$$f_Y(y) = 2\sqrt{1-y^2} \qquad -1 \le y \le 1$$

$$f(x,y) \neq f_X(x)\, f_Y(y) \; !$$

$$f(x,y) = \frac{1}{\pi} \qquad (x,y) \in C$$

Independence

$$X \perp\!\!\!\perp Y \quad \text{if}$$

$$f(x,y) = f_X(x)\, f_Y(y)$$

Exponential r.v.

$$X \qquad f_X = \lambda e^{-\lambda x} \qquad x > 0$$

$X \quad Y \qquad$ independent

$$Z = X + Y$$

$$\mathbb{P}(Z \leq z) = \mathbb{P}(X + Y \leq z) =$$

$$= \int_A f(x,y)\, dx\, dy =$$

$$A = \{ (x,y) \mid x + y \leq z \}$$

$$= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{z-x} f(x,y)\, dx\, dy =$$

$$= F(z)$$

$$f_Z(z) = \frac{d}{dz} F(z) = \int_{-\infty}^{\infty} f(x, z-x)\, dx$$

if X Y are indep.

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$

$$f_Z = f_X * f_Y$$

$$f_X(x) = \lambda e^{-\lambda x} \qquad x > 0$$

$$f_Y(y) = \mu e^{-\mu x} \qquad y > 0$$

$$f_Z(z) = \lambda\mu \int_0^z e^{-\lambda x} e^{-\mu(z-x)}\, dx$$

$\lambda = \mu$

$$= \lambda^2 \int_0^z e^{-\lambda x - \lambda(z-x)}\, dx =$$

$$= \lambda^2 z\, e^{-\lambda z}$$

$X_3 \qquad \lambda e^{-\lambda x}$

$Z + X_3 = Z_3$

$$f_3(z) = \lambda^3 \int_0^z \underbrace{x e^{-\lambda x}}_{Z} \underbrace{e^{-\lambda(z-x)}}_{X_3}\, dx$$

$$= \lambda^3 e^{-\lambda z} \int_0^z x = \lambda^3 e^{-\lambda z} \frac{z^2}{2}$$

0

If I have n exponential r.v.

$X_1 \quad X_2 \quad \dots \quad X_n$ i.i.d.

$$Z_n = X_1 + X_2 + \dots X_n$$

$$f_n(z) = \frac{\lambda^n z^{n-1}}{(n-1)!} e^{-\lambda z}$$

Prob that before time t I get n events.

$$\mathbb{P}\left(\sum_{i=1}^{n} X_i < t \ \& \ \sum_{i=1}^{n+1} X_i > t\right) =$$

$$\frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

Poissonian with par λt.

Conditional distribution

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$$

$$X \perp\!\!\!\perp Y \iff f_{Y|X}(y|x) = f_Y(y).$$